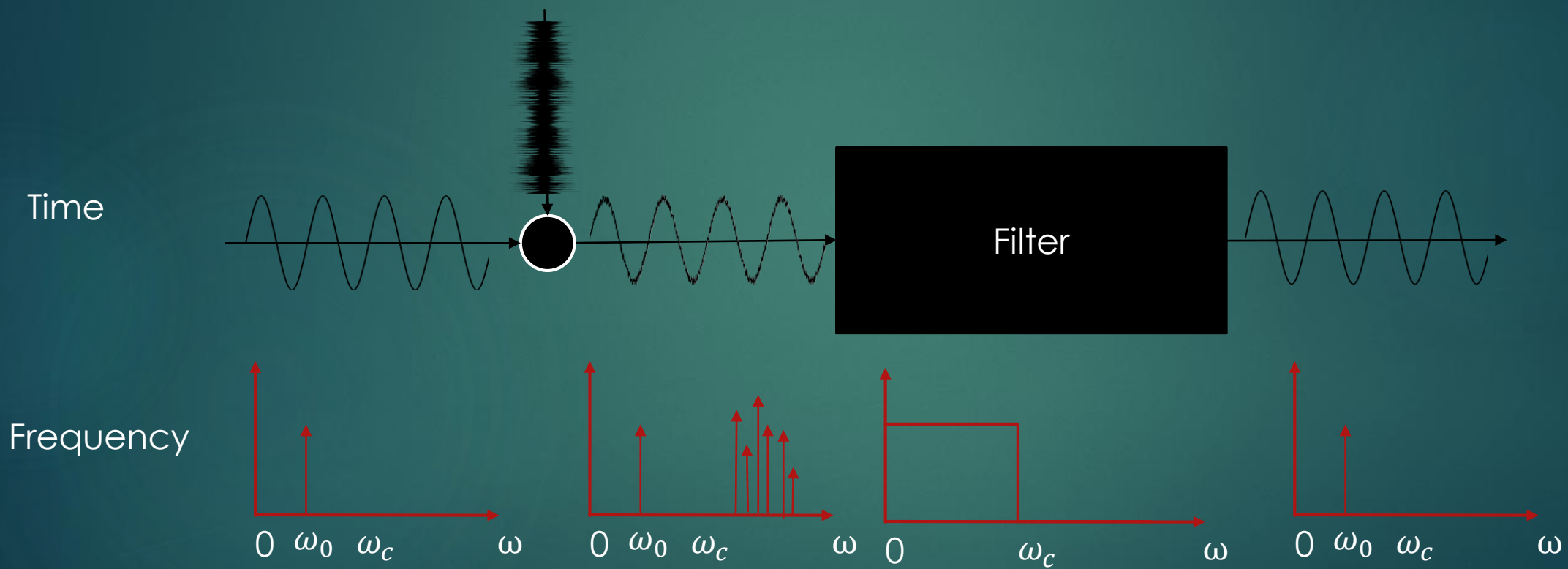




Computer Controlled Systems

DIGITAL FILTERS

How Filters Work

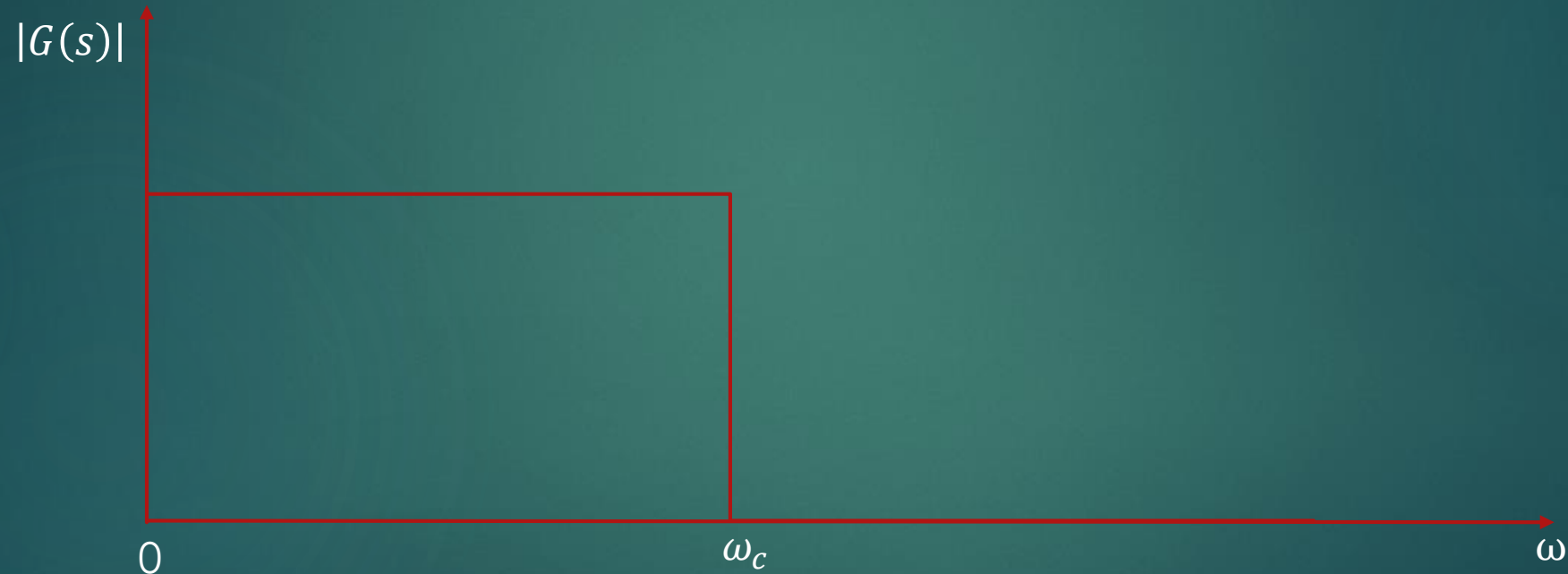


Filter Types

- ▶ Low-Pass Filter
- ▶ High-Pass Filter
- ▶ Band-Pass Filter
- ▶ Band-Stop Filter

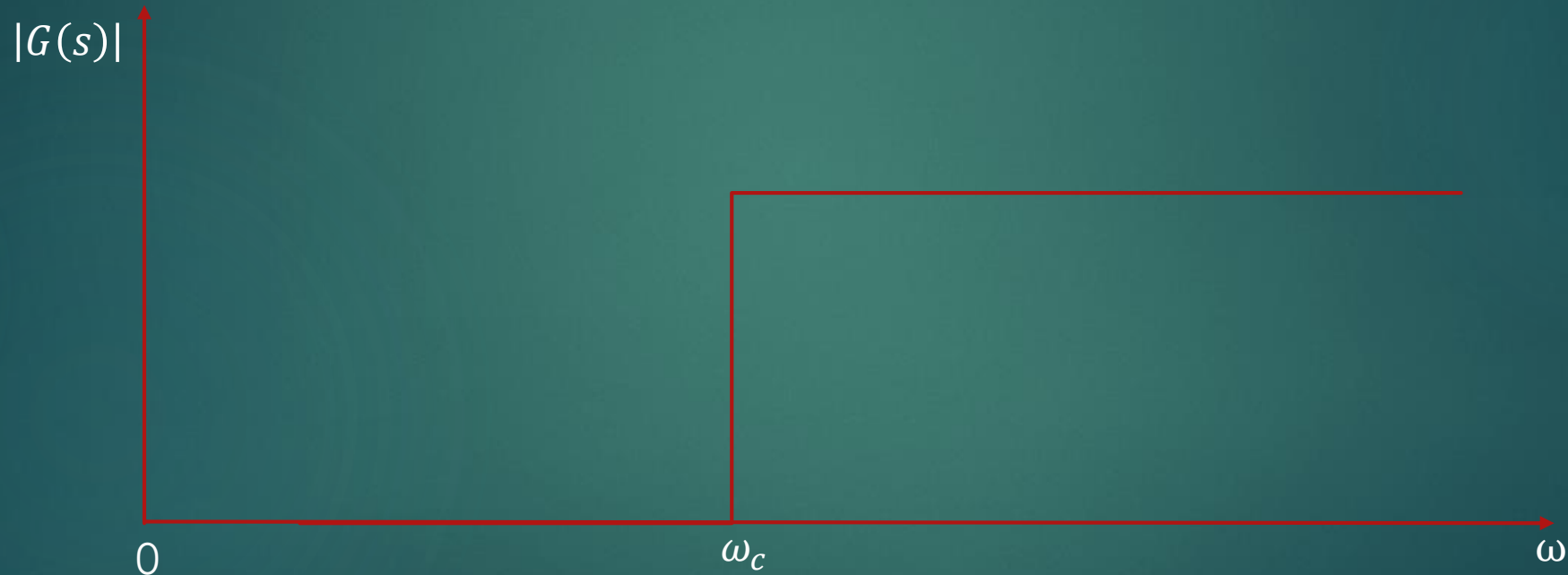
Low-Pass Filter (LPF)

Pass signals with frequency **lower** than the cut-off frequency (ω_c)



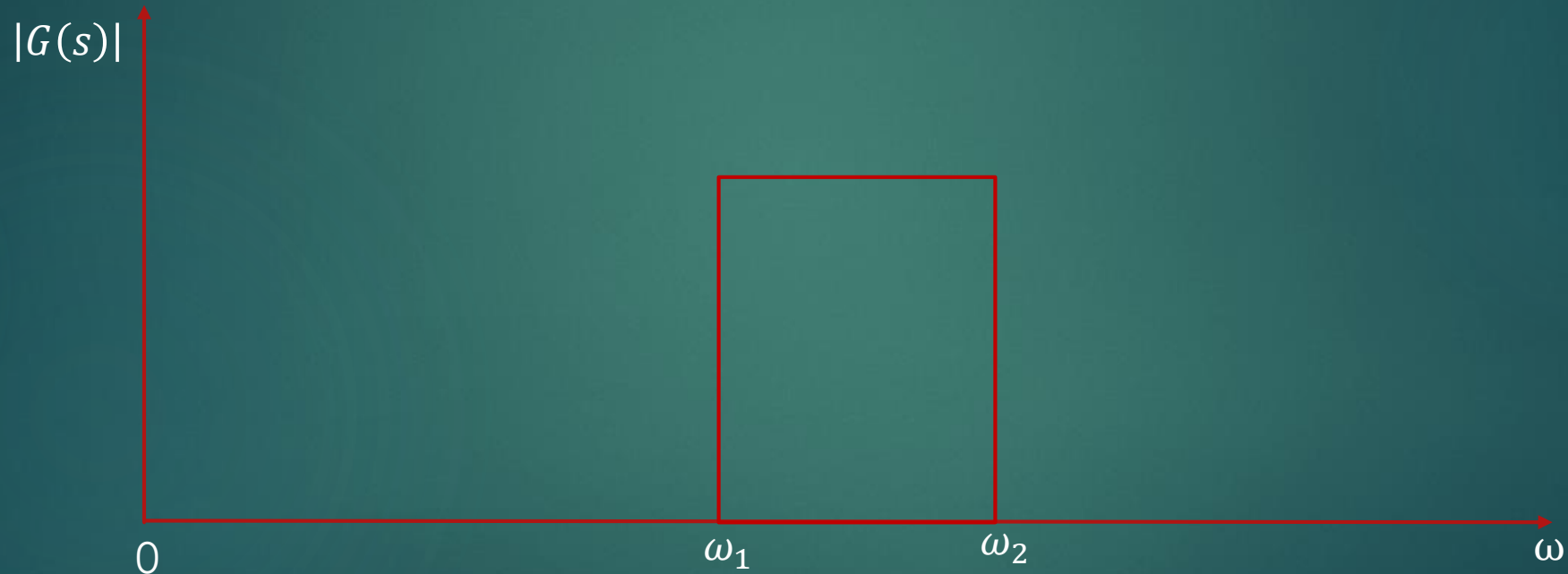
High-Pass Filter (HPF)

Pass signals with frequency **higher** than the cut-off frequency (ω_c)



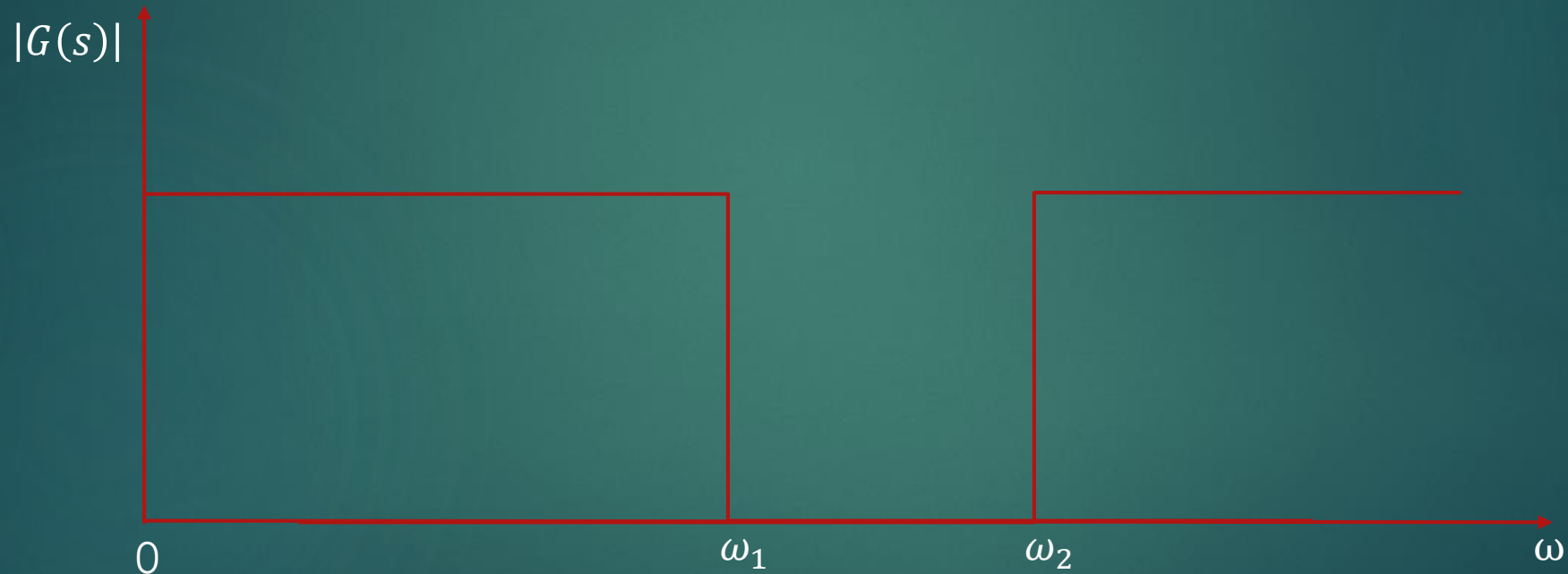
Band-Pass Filter (BPF)

Pass signals with frequency **between** two frequencies (ω_1 and ω_2)



Band-Stop Filter (BSF)

Stop signals with frequency **between** two frequencies (ω_1 and ω_2)



Binomial Low-Pass Filter

$$G(s) = \frac{\omega_c^n}{(s + \omega_c)^n}$$

For 1st order LPF:

$$G(s) = \frac{\omega_c}{s + \omega_c}$$

For 2nd order LPF:

$$G(s) = \frac{\omega_c^2}{(s + \omega_c)^2} = \frac{\omega_c^2}{s^2 + 2\omega_c s + \omega_c^2}$$

For 3rd order LPF:

$$G(s) = \frac{\omega_c^3}{(s + \omega_c)^3} = \frac{\omega_c^3}{s^3 + 3\omega_c s^2 + 3\omega_c^2 s + \omega_c^3}$$

Butterworth Low-Pass Filter

$$G(s) = \frac{\omega_c^n}{\prod_{k=0}^{n-1} (s - s_k)}$$

Where

$$s_k = \omega_c e^{j\frac{\pi}{2}\left(1 + \frac{2k+1}{n}\right)}$$

Butterworth Low-Pass Filter

$$s_k = \omega_c e^{\frac{j\pi}{2}\left(1 + \frac{2k+1}{n}\right)} ; k = 0, 1, \dots, n-1$$

For 1st order LPF ($n = 1$):

$$s_0 = \omega_c e^{j\pi} = -\omega_c$$

Substituting:

$$G(s) = \frac{\omega_c}{s + \omega_c}$$

Butterworth Low-Pass Filter

$$s_k = \omega_c e^{\frac{j\pi}{2}\left(1 + \frac{2k+1}{n}\right)} ; k = 0, 1, \dots, n-1$$

For 2nd order LPF ($n = 2$):

$$s_0 = \omega_c e^{j3\pi/4} = -\omega_c \left(\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \right)$$

$$s_1 = \omega_c e^{j7\pi/4} = \omega_c \left(\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \right)$$

$$G(s) = \frac{\omega_c^2}{\left(s + \frac{\omega_c}{\sqrt{2}}\right)^2 + \left(\frac{\omega_c}{\sqrt{2}}\right)^2}$$

$$G(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Butterworth Low-Pass Filter

$$s = \omega_c e^{\frac{j\pi}{2}\left(1 + \frac{2k+1}{n}\right)} ; k = 0, 1, \dots, n-1$$

For 3rd order LPF ($n = 3$):

$$s_0 = \omega_c e^{j2\pi/3} = -\omega_c(0.5 - j0.866)$$

$$s_1 = \omega_c e^{j\pi} = -\omega_c$$

$$s_2 = \omega_c e^{j4\pi/3} = -\omega_c(0.5 + j0.866)$$

$$G(s) = \frac{\omega_c^3}{(s + 1)[(s + 0.5\omega_c)^2 + (0.866^2)]}$$

$$G(s) = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$$

Bessel Low-Pass Filter

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)}$$

Where

$$\theta_n(x) = \sum_{k=0}^n a_k x^k$$

And

$$a_k = \frac{(2n-k)!}{2^{n-k} k! (n-k)!}$$

And

$$\theta_n(0) = \lim_{x \rightarrow 0} \theta_n(x)$$

Bessel Low-Pass Filter

For 1st order LPF ($n = 1$):

$$a_k = \frac{(2n - k)!}{2^{n-k} k! (n - k)!} = \frac{(2 - k)!}{2^{1-k} k! (1 - k)!} \Rightarrow a_0 = 1, a_1 = 1$$

$$\theta_1\left(\frac{s}{\omega_c}\right) = \sum_{k=0}^1 a_k \left(\frac{s}{\omega_c}\right)^k = 1 + \frac{s}{\omega_c}$$

$$\theta_1(0) = \lim_{x \rightarrow 0} \left[\sum_{k=0}^1 a_k x^k \right] = \lim_{x \rightarrow 0} [1 + x] = 1$$

Substituting:

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)} = \frac{1}{1 + \frac{s}{\omega_c}} = \frac{\omega_c}{s + \omega_c}$$

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)}$$

$$\theta_n(x) = \sum_{k=0}^n a_k x^k$$

$$a_k = \frac{(2n - k)!}{2^{n-k} k! (n - k)!}$$

$$\theta_n(0) = \lim_{x \rightarrow 0} \theta_n(x)$$

Bessel Low-Pass Filter

For 2nd order LPF ($n = 2$):

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!} = \frac{(4-k)!}{2^{2-k}k!(2-k)!} \Rightarrow a_0 = 3, a_1 = 3, a_2 = 1$$

$$\theta_2\left(\frac{s}{\omega_c}\right) = \sum_{k=0}^2 a_k \left(\frac{s}{\omega_c}\right)^k = 3 + \frac{3s}{\omega_c} + \frac{s^2}{\omega_c^2}$$

$$\theta_2(0) = \lim_{x \rightarrow 0} \left[\sum_{k=0}^2 a_k x^k \right] = \lim_{x \rightarrow 0} [3 + 3x + x^2] = 3$$

Substituting:

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)} = \frac{3}{3 + \frac{3s}{\omega_c} + \frac{s^2}{\omega_c^2}} = \frac{3\omega_c^2}{s^2 + 3\omega_c s + 3\omega_c^2}$$

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)}$$

$$\theta_n(x) = \sum_{k=0}^n a_k x^k$$

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!}$$

$$\theta_n(0) = \lim_{x \rightarrow 0} \theta_n(x)$$

Bessel Low-Pass Filter

For 3rd order LPF ($n = 3$):

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!} = \frac{(6-k)!}{2^{3-k}k!(3-k)!} \Rightarrow a_0 = 15, a_1 = 15, a_2 = 6, a_3 = 1$$

$$\theta_3\left(\frac{s}{\omega_c}\right) = \sum_{k=0}^3 a_k \left(\frac{s}{\omega_c}\right)^k = 15 + \frac{15s}{\omega_c} + \frac{6s^2}{\omega_c^2} + \frac{s^3}{\omega_c^3}$$

$$\theta_3(0) = \lim_{x \rightarrow 0} \left[\sum_{k=0}^3 a_k x^k \right] = \lim_{x \rightarrow 0} [15 + 15x + 6x^2 + x^3] = 15$$

Substituting:

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)} = \frac{15}{15 + \frac{15s}{\omega_c} + \frac{6s^2}{\omega_c^2} + \frac{s^3}{\omega_c^3}} = \frac{15\omega_c^3}{s^3 + 6\omega_c s^2 + 15\omega_c^2 s + 15}$$

$$G(s) = \frac{\theta_n(0)}{\theta_n\left(\frac{s}{\omega_c}\right)}$$

$$\theta_n(x) = \sum_{k=0}^n a_k x^k$$

$$a_k = \frac{(2n-k)!}{2^{n-k}k!(n-k)!}$$

$$\theta_n(0) = \lim_{x \rightarrow 0} \theta_n(x)$$

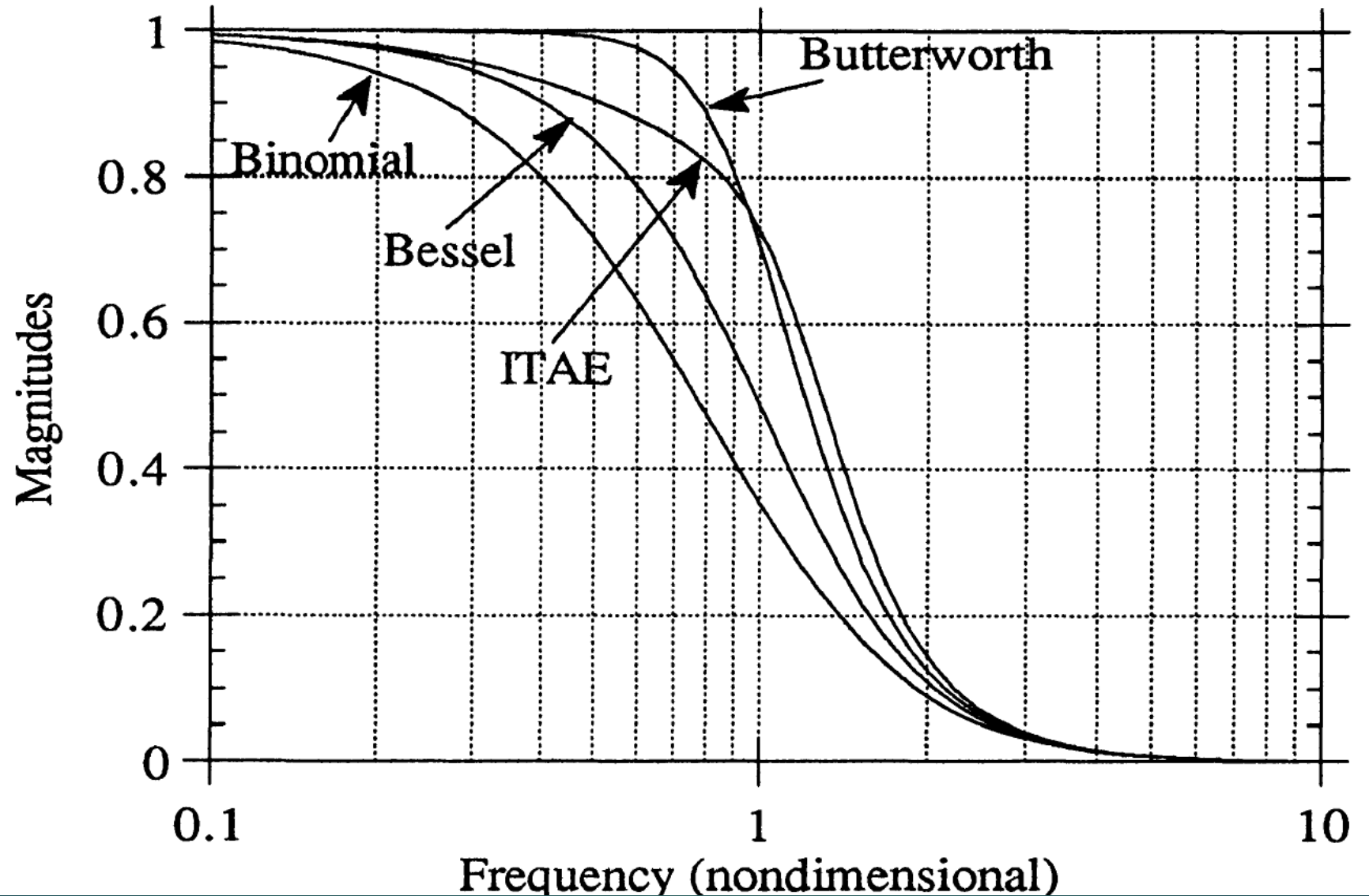


Figure source: Computer Controlled Systems – Theory and Applications (textbook)

High-Pass Filter Design

$$G_{HPF}(s) = G_{LPF}\left(\frac{1}{s}\right)$$

For 1st order LPF:

$$G_{LPF}(s) = \frac{\omega_c}{s + \omega_c}$$

$$G_{HPF}(s) = G_{LPF}\left(\frac{1}{s}\right) = \frac{\omega_c}{\frac{1}{s} + \omega_c}$$

$$G_{HPF}(s) = \frac{s}{s + \frac{1}{\omega_c}}$$

High-Pass Filter Design

For 2nd order LPF:

$$G_{LPF}(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

$$G_{HPF}(s) = G_{LPF}\left(\frac{1}{s}\right) = \frac{\omega_c^2}{\frac{1}{s^2} + \frac{\sqrt{2}\omega_c}{s} + \omega_c^2}$$

$$G_{HPF}(s) = \frac{s^2}{s^2 + \frac{\sqrt{2}}{\omega_c}s + \frac{1}{\omega_c^2}}$$

Band-Pass Filter Design

$$G_{BPF}(s) = G_{LPF}\left(\frac{s^2 + \omega_c^2}{Bs}\right)$$

Where

$$\omega_c = \omega_1 \omega_2 \text{ and } B = (\omega_2 - \omega_1)$$

For 1st order LPF:

$$G_{LPF}(s) = \frac{\omega_c}{s + \omega_c}$$

$$G_{BPF}(s) = G_{LPF}\left(\frac{s^2 + \omega_c^2}{Bs}\right) = \frac{\omega_c}{\frac{s^2 + \omega_c^2}{Bs} + \omega_c}$$

We get 2nd order BPF:

$$G_{BPF}(s) = \frac{B\omega_c s}{s^2 + B\omega_c s + \omega_c^2}$$

Band-Stop Filter Design

$$G_{BSF}(s) = G_{HPF}\left(\frac{s^2 + \omega_c^2}{Bs}\right)$$

Where

$$\omega_c = \omega_1\omega_2 \text{ and } B = (\omega_2 - \omega_1)$$

For 1st order HPF:

$$G_{HPF}(s) = \frac{s}{s + \frac{1}{\omega_c}}$$

$$G_{BSF}(s) = G_{HPF}\left(\frac{s^2 + \omega_c^2}{Bs}\right) = \frac{\frac{s^2 + \omega_c^2}{Bs}}{\frac{s^2 + \omega_c^2}{Bs} + 1/\omega_c}$$

We get 2nd order BSF:

$$G_{BSF}(s) = \frac{s^2 + \omega_c^2}{s^2 + \frac{B}{\omega_c}s + \omega_c^2}$$

Digital Filter Design

- ▶ Impulse Invariance Method

$$G(z) = TZ[G(s)]$$

- ▶ Step Invariance Method

$$G(z) = (1 - z^{-1})Z\left[\frac{G(s)}{s}\right]$$

- ▶ Bilinear Transformation Method

$$G(z) = G(s) \Big|_{s=\frac{z-1}{z+1}}$$

$$\omega_c = \tan\left(\frac{\omega_d T}{2}\right)$$

Where ω_d is the digital cut-off frequency and T is the sampling time

Digital Filter Design - Example

$$G(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Cut-off Frequency is 50 rad/sec and sampling frequency is 60Hz

Digital Filter Design - Example

$$G(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Cut-off Frequency is 50rad/sec and sampling frequency is 60Hz

$$G(s) = \frac{2500}{s^2 + 70.7s + 2500}$$

► Impulse Invariance Method:

$$G(z) = TZ[G(s)]$$

$$G(z) = TZ[G(s)] = \frac{1}{60} Z \left[\frac{2500}{s^2 + 70.7s + 2500} \right] = \frac{0.363z}{z^2 - 0.922z + 0.306}$$

Digital Filter Design - Example

$$G(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Cut-off Frequency is 50rad/sec and sampling frequency is 60Hz

$$G(s) = \frac{2500}{s^2 + 70.7s + 2500}$$

► Step Invariance Method:

$$G(z) = (1 - z^{-1}) Z \left[\frac{G(s)}{s} \right]$$

$$G(z) = (1 - z^{-1}) Z \left[\frac{G(s)}{s} \right] = (1 - z^{-1}) Z \left[\frac{2500}{s(s^2 + 70.7s + 2500)} \right] = \frac{0.2306z + 0.1549}{z^2 - 0.922z + 0.3077}$$

Digital Filter Design - Example

$$G(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Cut-off Frequency is 50rad/sec and sampling frequency is 60Hz

► Bilinear Transformation Method:

$$\omega_c = \tan\left(\frac{\omega_d T}{2}\right)$$
$$G(z) = G(s) \Big|_{s=\frac{z-1}{z+1}}$$

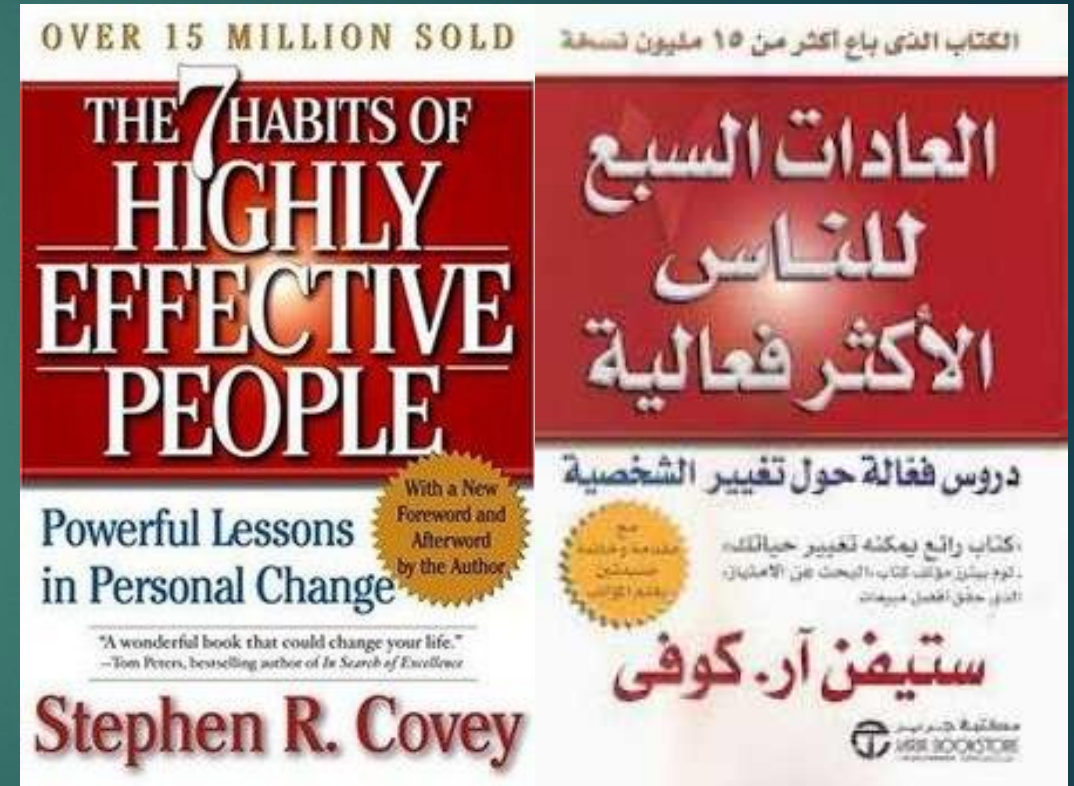
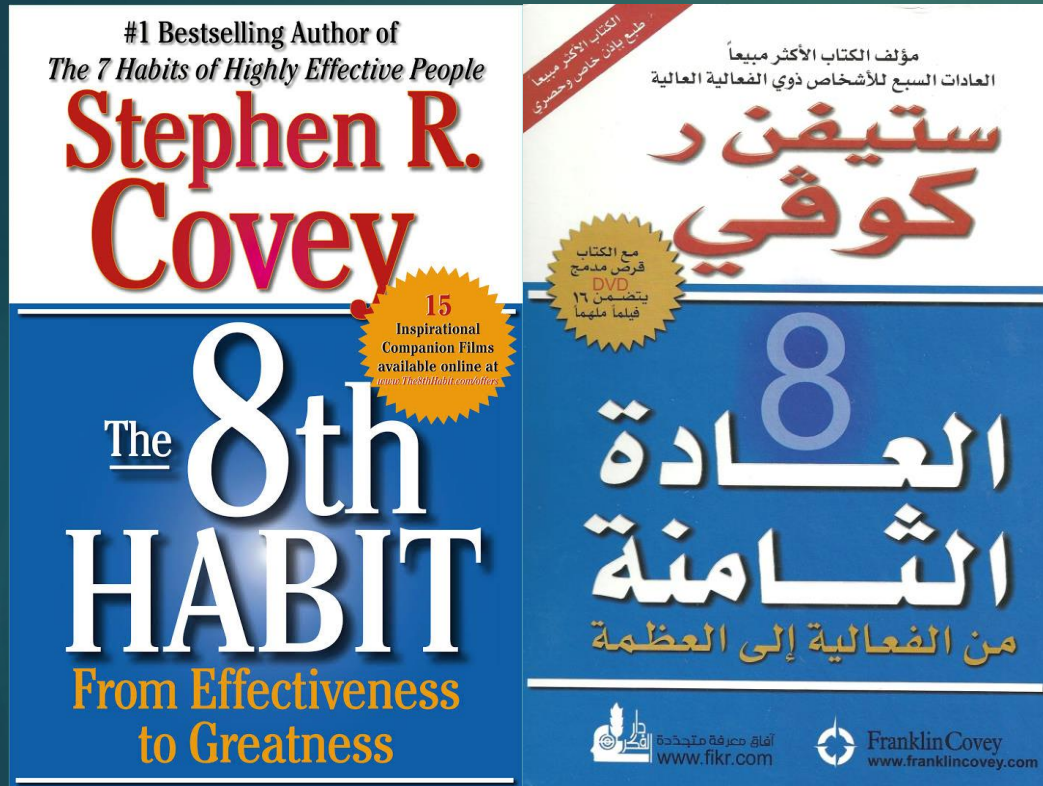
$$\omega_c = \tan\left(\frac{\omega_d T}{2}\right) = \tan\left(\frac{50}{2 * 60}\right) = 0.4426$$
$$G(s) = \frac{0.1959}{s^2 + 0.2658s + 0.1959}$$

$$G(z) = G(s) \Big|_{s=\frac{z-1}{z+1}} = \frac{0.1959}{\left(\frac{z-1}{z+1}\right)^2 + 0.2658\left(\frac{z-1}{z+1}\right) + 0.1959} = \frac{0.1075z^2 + 0.215z + 0.1075}{z^2 - 0.8828z + 0.3129}$$

Reading List

Sections 5.2, 5.3, and 5.4

Non-academic Reading



RAMADAN
Kareem

