



Computer Controlled Systems

STATE SPACE MODELS

Prior Knowledge

- ▶ Transfer Functions, Block Diagrams
 - ▶ Covered in CSE 313 (Automatic Control)
 - ▶ Reviewed in sheet 2
 - ▶ Read section 2.6
- ▶ Z-transform, Inverse Z-transform
 - ▶ Covered in CSE 416 (Digital Control)
 - ▶ Reviewed in sheet 3
 - ▶ Read section 3.5

Prior Knowledge

- ▶ Sampled Block Diagrams
 - ▶ Covered in sheet 4
 - ▶ Read section 3.7
- ▶ Stability in discrete systems
 - ▶ Covered in Sheet 5
 - ▶ Read section 3.8

State Space Models

- ▶ One more way to represent a control system
- ▶ Defines the system in terms of *system states*
- ▶ Can be used for MIMO systems

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

State Space Models

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

$\mathbf{u}$: System input(s)

$\mathbf{y}$: System Output(s)

$\mathbf{x}$: System states

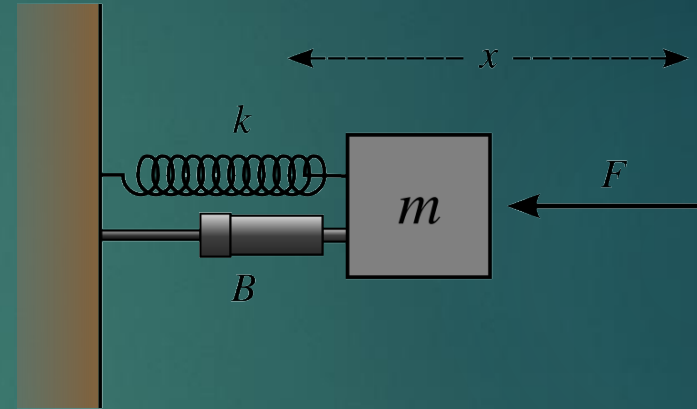
$\dot{\mathbf{x}}$: System state derivatives

Mass-Spring System

Input : $u = F$

Output : $y = x$

States: ??



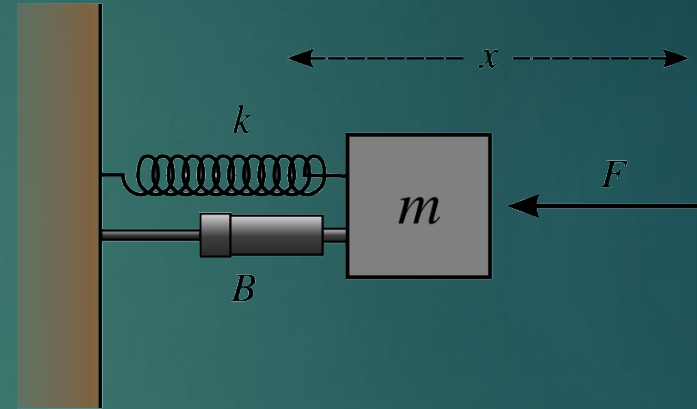
$$m\ddot{x} = F - b\dot{x} - kx$$

Mass-Spring System

Input : $u = F$

Output : $y = x$

States : $x_1 = x, x_2 = \dot{x}$



$$m\ddot{x} = F - b\dot{x} - kx$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}u - \frac{b}{m}x_2 - \frac{k}{m}x_1 \\ y &= x_1 \end{aligned}$$

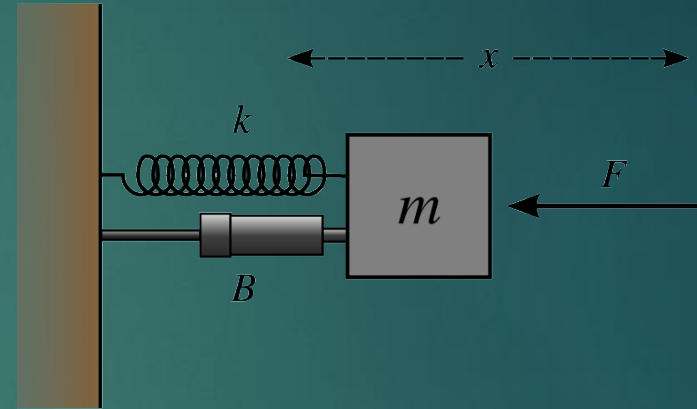
Mass-Spring System

Input : $u = F$

Output : $y = x$

States : $x_1 = x, x_2 = \dot{x}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$m\ddot{x} = F - b\dot{x} - kx$$

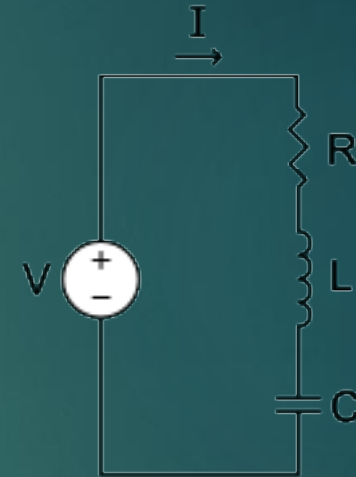
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}u - \frac{b}{m}x_2 - \frac{k}{m}x_1 \\ y &= x_1 \end{aligned}$$

RLC Circuit

Input : $u = V$

Outputs : $y_1 = V_R$, $y_2 = V_C$

States : ??



$$V_R = RI$$

$$V_L = L\dot{I}$$

$$I = C\dot{V}_C$$

$$V = V_R + V_L + V_C$$

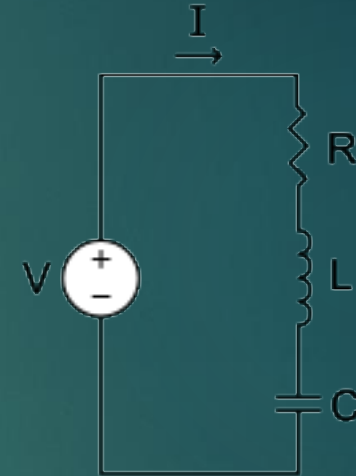
RLC Circuit

Input : $u = V$

Outputs : $y_1 = V_R$, $y_2 = V_C$

States : $x_1 = I$, $x_2 = V_C$

$$\begin{aligned}\dot{I} &= \frac{1}{L}V - \frac{R}{L}I - \frac{1}{L}V_C \\ \dot{V}_C &= \frac{1}{C}I \\ V_R &= RI\end{aligned}$$



$$\begin{aligned}V_R &= RI \\ V_L &= L\dot{I} \\ I &= C\dot{V}_C \\ V &= V_R + V_L + V_C\end{aligned}$$

RLC Circuit

Input : $u = V$

Outputs : $y_1 = V_R$, $y_2 = V_C$

States : $x_1 = I$, $x_2 = V_C$

$$\dot{x}_1 = \frac{1}{L}u - \frac{R}{L}x_1 - \frac{1}{L}x_2$$

$$\dot{x}_2 = \frac{1}{C}x_1$$

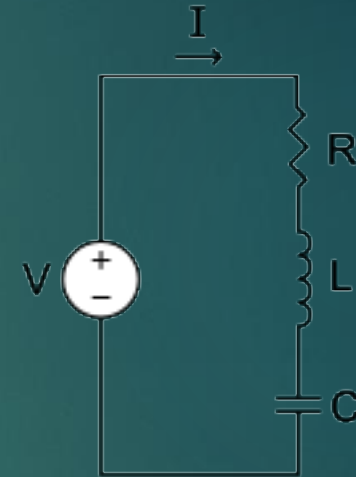
$$y_1 = Rx_1$$

$$y_2 = x_2$$

$$\dot{I} = \frac{1}{L}V - \frac{R}{L}I - \frac{1}{L}V_C$$

$$\dot{V}_C = \frac{1}{C}I$$

$$V_R = RI$$



$$V_R = RI$$

$$V_L = L\dot{I}$$

$$I = C\dot{V}_C$$

$$V = V_R + V_L + V_C$$

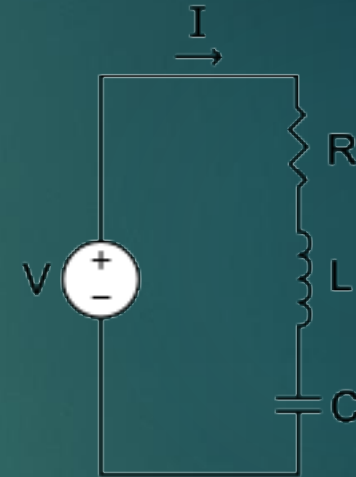
RLC Circuit

Input : $u = V$

Outputs : $y_1 = V_R$, $y_2 = V_C$

States : $x_1 = I$, $x_2 = V_C$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$V_R = RI$$

$$V_L = L\dot{I}$$

$$I = C\dot{V}_C$$

$$V = V_R + V_L + V_C$$

State Space to Transfer Function

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$sX(s) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1}BU(s)$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$Y(s) = CX(s) + DU(s)$$

$$Y(s) = [C(sI - A)^{-1}B + D] U(s)$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Example – Mass-Spring System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}$$
$$C = [1 \quad 0], D = 0$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} = \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + \frac{b}{m}s + \frac{k}{m}} [1 \quad 0] \begin{bmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + 0$$

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

Discrete State Space Models

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{aligned}$$

$$\frac{Y(z)}{U(z)} = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Controllability & Observability

- ▶ A system is controllable
 - ▶ If the state vector can be determined from the input vector
 - ▶ i.e., iff the rank of the *controllability matrix* is n :

$$\text{Rank}[B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] = n$$

- ▶ A system is observable
 - ▶ If the state vector can be determined from the output vector
 - ▶ i.e., iff the rank of the *observability matrix* is n :

$$\text{Rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Example – Mass-Spring System

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, C = [1 \quad 0]$$

$$AB = \begin{bmatrix} \frac{1}{m} \\ -\frac{b}{m^2} \end{bmatrix} \Rightarrow [B \ AB] = \begin{bmatrix} 0 & \frac{1}{m} \\ \frac{1}{m} & -\frac{b}{m^2} \end{bmatrix} \Rightarrow |B \ AB| = -\frac{1}{m^2} \neq 0 \Rightarrow \text{Rank} = 2 \Rightarrow \text{Controllable}$$

$$CA = [0 \quad 1] \Rightarrow \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{vmatrix} C \\ CA \end{vmatrix} = 1 \neq 0 \Rightarrow \text{Rank} = 2 \Rightarrow \text{Observable}$$

Quiz

1. Find the transfer functions of the RLC system given the state space model
2. Determine if the RLC system is controllable and observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Reading List

Sections 4.1, 4.2-2, 4.3-2, 4.6

Non-academic Reading

