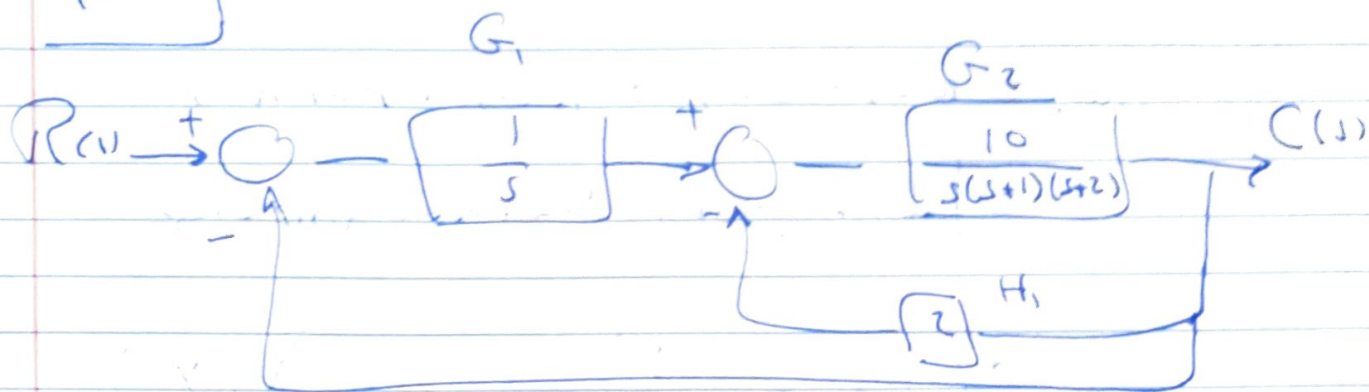


Q1-a)



$$F(s) = \frac{G_2(s)}{1 + G_2(s)H(s)} = \frac{10}{s(s+1)(s+2) + 20}$$

$$= \frac{10}{s^3 + 3s^2 + 2s + 20}$$

$$F_1(s) G_1(s) = \frac{10}{s^4 + 3s^3 + 2s^2 + 20s}$$

$$F(s) = \frac{C(s)}{R(s)} = \frac{F_1(s)G_1(s)}{1 + F_1(s)G_1(s)} = \frac{10}{s^4 + 3s^3 + 2s^2 + 20s + 10}$$

≡

(2)

Q1-b)

$$E(s) = R(s) - C(s) = R(s) - F(s)R(s)$$

$$= [1 - F(s)] R(s) \leftarrow \text{unit step}$$

$$= \left(1 - \frac{10}{s^4 + 3s^3 + 2s^2 + 20s + 10} \right) \left(\frac{1}{s} \right)$$

$$= \frac{s^4 + 3s^3 + 2s^2 + 20s + \cancel{10} - \cancel{10}}{s(s^4 + 3s^3 + 2s^2 + 20s + 10)}$$

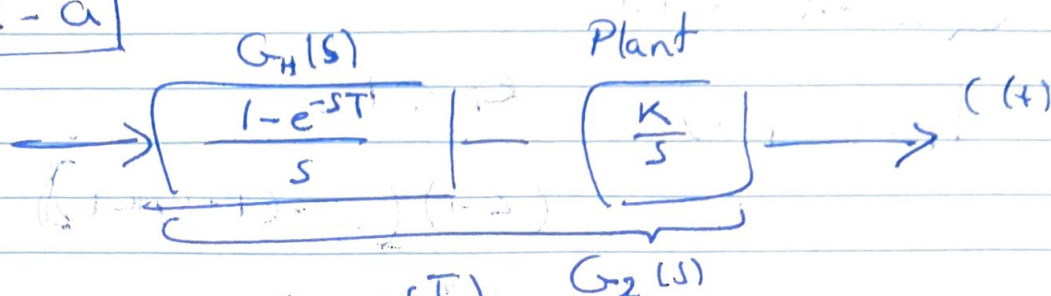
$$e_{ss} = \lim_{s \rightarrow 0} s E(s) =$$

$$= \lim_{s \rightarrow 0} \frac{s^4 + 3s^3 + 2s^2 + 20s}{s^4 + 3s^3 + 2s^2 + 20s + 10}$$

$$= 0$$

✱

Q2-a

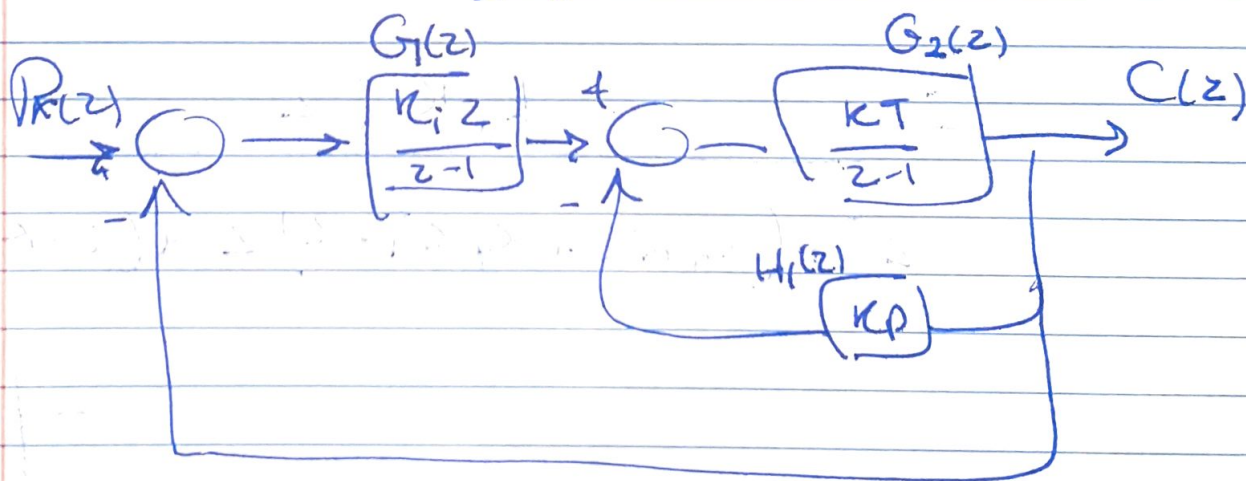


$$G_2(s) = \frac{K(1 - e^{-sT})}{s^2}$$

$$G_2(z) = \mathcal{Z}\{G_2(s)\} = \mathcal{Z}\left\{\frac{K(1 - e^{-sT})}{s^2}\right\}$$

from table $\Rightarrow (1 - z^{-1}) \mathcal{Z}\left\{\frac{K}{s^2}\right\} = \frac{z-1}{z} \frac{KT}{(z-1)^2}$

$$\therefore G_2(z) = \frac{KT}{z-1}$$



$$F_1(z) = \frac{G_2(z)}{1 + G_2(z)H_1(z)} = \frac{KT/(z-1)}{1 + K_p KT/(z-1)}$$

$$= \frac{KT}{z-1 + K_p KT}$$

Note: $D(z) = 1$

(4)

Q2. (contd)

$$F_1(z)G_1(z) = \frac{k_i K T z}{(z-1)[z-(1-k_p K T)]}$$

$$= \frac{k_i K T z}{z^2 + (k_p K T - 2)z + (1 - k_p K T)}$$

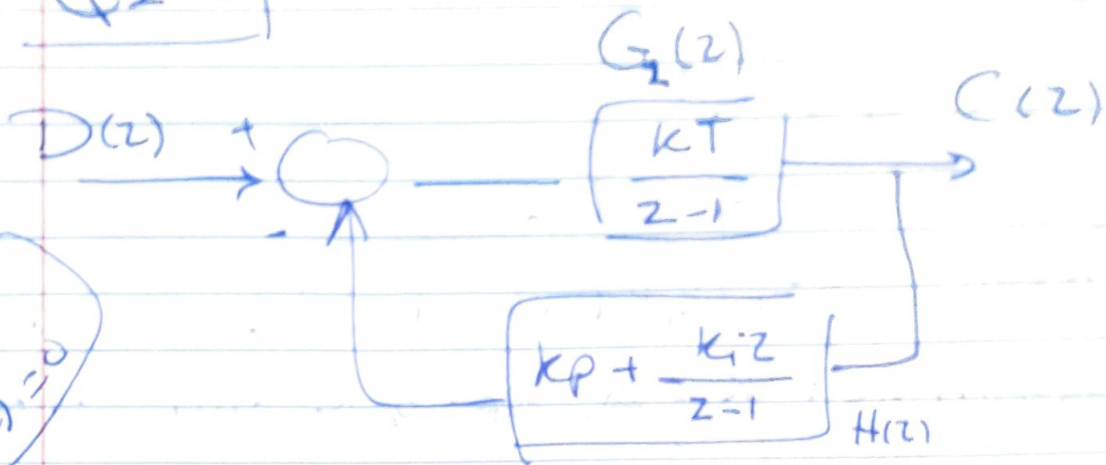
$$\frac{C(z)}{R(z)} = \frac{F_1(z)G_1(z)}{1 + F_1(z)G_1(z)}$$

$$= \frac{k_i K T z}{z^2 + (k_p K T - 2)z + (1 - k_p K T) + k_i K T z}$$

$$= \frac{k_i K T z}{z^2 + (k_i K T + k_p K T - 2)z + (1 - k_p K T)}$$

#

Q2-b

Note
 $R(z) = 0$

$$\frac{C(z)}{D(z)} = \frac{G_2(z)}{1 + G_2(z)H(z)}$$

$$= \frac{KT/(z-1)}{1 + \left(\frac{KT}{z-1}\right) \left(K_p + \frac{K_i z}{z-1}\right)}$$

$$= \frac{KT(z-1)}{z^2 + (KK_pT + KK_iT - 2)z + (1 - KK_pT)}$$

#

Q2-c)

from Q2-a

$$\frac{C(z)}{R(z)} = \frac{K_i K_T z}{z^2 + (K_i K_T + K_p K_T - 2)z + (1 - K_p K_T)}$$

Substitute: $K = 20$, $K_p = 5$, $K_i = 1.2$, $T = 0.005$

$$\frac{C(z)}{R(z)} = \frac{0.125 z}{z^2 - 1.375 z + 0.5}$$

for unit-step input $R(z) = \frac{z}{z-1}$

$$Y(z) = \frac{0.125 z^2}{(z-1)(z^2 - 1.375 z + 0.5)}$$

Using Partial Fractions:

$$\frac{C(z)}{z} = \frac{A}{z-1} + \frac{Bz + C}{z^2 - 1.375z + 0.5}$$

Solving for A, B, and C:

$$\begin{aligned} C(z) &= \frac{z}{z-1} - \frac{z^2 - 0.5z}{z^2 - 1.375z + 0.5} \\ &= \frac{z}{z-1} - \frac{z^2 - 0.6875z}{z^2 - 1.375z + 0.5} + 1.1358 \frac{0.16537z}{z^2 - 1.375z + 0.5} \end{aligned}$$

$\swarrow$ $\sin + \cos$

(7)

Q2-c) cont'd

From Z-transform tables:

$$c(k) = 1 - e^{-0.346} (\cos 0.236k + 1.1586 \sin 0.236k)$$

for $k = 0, 1, \dots$

#

Q2-d) $n=2$

$$F(z) = z^2 + (k_i k_p T + k_p k_i T - 2)z + (1 - k_p k_i T)$$

$$a_0 = 1 - k_p k_i T, a_1 = (k_i k_p T + k_p k_i T - 2), a_2 = 1$$

$$\left. \begin{array}{l} 1) a_2 + a_1 + a_0 > 0 \\ 2) a_2 - a_1 + a_0 > 0 \\ 3) |a_0| < a_2 \end{array} \right\} \text{3 conditions}$$

$$1) 1 + k_p k_i T + k_p k_i T - 2 + 1 - k_p k_i T = k_p k_i T > 0$$

$$2) 1 - k_p k_i T - k_p k_i T + 2 + 1 - k_p k_i T = 4 - 2k_p k_i T - k_p k_i T > 0$$

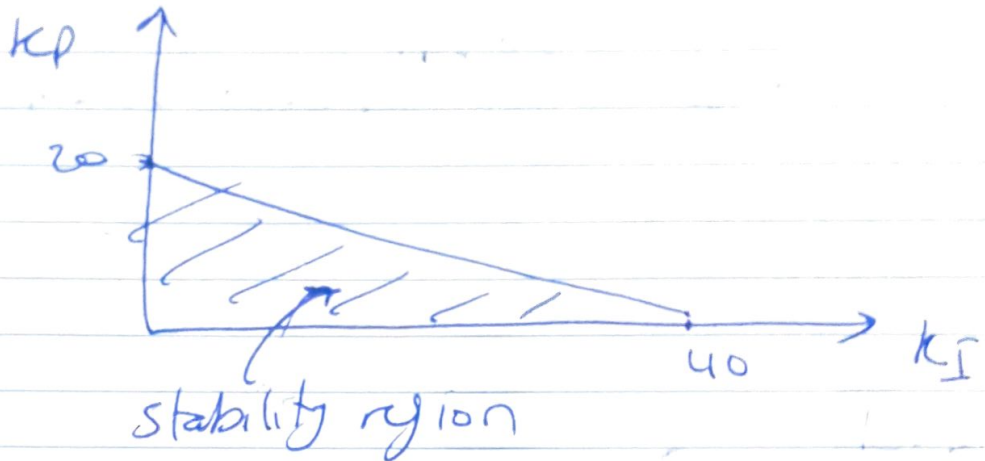
$$3) |1 - k_p k_i T| < 1 \Rightarrow 0 < k_p k_i T < 2$$

$$\text{Using } K=20, T=0.005 \Rightarrow \text{OK}$$

8

Q2-d) contd

$$k_i > 0, \quad k_p < \frac{40 - k_i}{2}, \quad 0 < k_p < 20$$



Q3-a)

$$V_R = RI, V_L = L\dot{I}, I = C\dot{V}_C, V = V_R + V_L + V_C$$

Selecting states as $x_1 = I$ and $x_2 = V_C$

Given: $u = V, y_1 = V_R, y_2 = V_C$ (or vice versa)

Rewriting in terms of states, input, and output:

$$V = V_R + V_L + V_C \Rightarrow u = R x_1 + L \dot{x}_1 + x_2$$

$$\Rightarrow \dot{x}_1 = \frac{1}{L} (u - R x_1 - x_2)$$

$$I = C\dot{V}_C \Rightarrow \dot{x}_1 = C \dot{x}_2 \Rightarrow \dot{x}_2 = \frac{1}{C} x_1$$

$$V_R = RI \Rightarrow y_1 = R x_1$$

$$y_2 = x_2 \quad (x_2 = V_C)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_B u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}}_C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$D = 0$$

Q3-b)

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

$$A = \begin{bmatrix} -R/L & -1/L \\ 1/L & 0 \end{bmatrix}, B = \begin{bmatrix} 1/L \\ 0 \end{bmatrix}, C = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}, D = 0$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -R/L & -1/L \\ 1/L & 0 \end{bmatrix} = \begin{bmatrix} s + R/L & 1/L \\ -1/L & s \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{s(s + R/L) + 1/L} \begin{bmatrix} s & -1/L \\ 1/L & s + R/L \end{bmatrix}$$

$$(sI - A)^{-1}B = \begin{bmatrix} s & -1/L \\ 1/L & s + R/L \end{bmatrix} \begin{bmatrix} 1/L \\ 0 \end{bmatrix} = \begin{bmatrix} s/L \\ 1/L \end{bmatrix}$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s/L \\ 1/L \end{bmatrix}$$

$$= \begin{bmatrix} Rs/L \\ 1/L \end{bmatrix}$$

$$\therefore \frac{Y_1(s)}{U(s)} = \frac{Rs/L}{s^2 + \frac{R}{L}s + 1/L}$$

$$\frac{Y_2(s)}{U(s)} = \frac{1/L}{s^2 + \frac{R}{L}s + 1/L}$$

≠

Q3-c $n=2$

$$[B \ AB] =$$

$$\begin{bmatrix} 1/L & -R/L^2 \\ 0 & 1/LC \end{bmatrix}$$

$$|B \ AB| = 1/L^2 C \neq 0 \Rightarrow \text{rank} = 2$$

$\Rightarrow$ Controllable

#

Q3-d $n=2$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 1/L \\ -R/L & -1/L \\ 1/C & 0 \end{bmatrix}$$

At least two rows are independent

$$\text{e.g. } \begin{vmatrix} -R/L & -1/L \\ 1/C & 0 \end{vmatrix} \neq 0 \Rightarrow \text{rank} = 2$$

$\therefore$ Observable

Q4-a)

$$\begin{aligned} n=2 \Rightarrow G(s) &= \frac{\omega_c^2}{(s+\omega_c)^2} = \frac{2500}{(s+50)^2} \\ \text{binomial LPF} & \\ &= \frac{2500}{s^2 + 100s + 2500} \quad \# \end{aligned}$$

Q4-b)

$n=3$, Butterworth, $\omega_c = 50$

$$s_0 = \omega_c e^{j2\pi/3} = -\omega_c \left(0.5 - j\frac{\sqrt{3}}{2} \right)$$

$$s_1 = \omega_c e^{j4\pi/3} = -\omega_c$$

$$s_2 = 50 e^{j4\pi/3} = -\omega_c \left(0.5 + j\frac{\sqrt{3}}{2} \right)$$

$$\begin{aligned} G(s)_{\text{LPF}} &= \frac{\omega_c^3}{(s+\omega_c) \left[(s+0.5\omega_c)^2 + \left(\frac{\sqrt{3}}{2}\omega_c \right)^2 \right]} \\ &= \frac{\omega_c^3}{(s+\omega_c) (s^2 + \omega_c s + \omega_c^2)} \\ &= \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3} \end{aligned}$$

$$\begin{aligned} G(s)_{\text{HPF}} &= G\left(\frac{1}{s}\right) = \frac{\omega_c^3}{\frac{1}{s^3} + 2\omega_c \frac{1}{s^2} + 2\omega_c^2 \frac{1}{s} + \omega_c^3} \\ &= \frac{s^3}{s^3 + \frac{2}{\omega_c} s^2 + \frac{2}{\omega_c^2} s + \frac{1}{\omega_c^3}} \quad \# \end{aligned}$$

Q4-c

$$G_{\text{HPF}}(z) \rightarrow G_{\text{HPF}}(s) \Big|_{s = \frac{z-1}{z+1}}$$

$$\omega_c = \tan\left(\frac{\omega_d T}{2}\right) = \tan\left(\frac{50}{2 \times 60}\right) = 0.4426$$

from 4-b:

$$G_{\text{HPF}}(s) = \frac{s^3}{s^3 + \frac{2}{\omega_c} s^2 + \frac{2}{\omega_c^2} s + \frac{1}{\omega_c^3}}$$

$$\therefore G_{\text{HPF}}(z) = \frac{\left(\frac{z-1}{z+1}\right)^3}{\left(\frac{z-1}{z+1}\right)^3 + \frac{2}{\omega_c} \left(\frac{z-1}{z+1}\right)^2 + \frac{2}{\omega_c^2} \left(\frac{z-1}{z+1}\right) + \frac{1}{\omega_c^3}}$$

$$\times \frac{\omega_c^3}{\omega_c^3} = \frac{\omega_c^3 (z-1)^3}{\frac{1}{\omega_c^3} (z+1)^3 + \frac{2}{\omega_c} \omega_c^3 (z-1)(z+1)^2 + \frac{2\omega_c^2}{\omega_c^2} (z-1)^2(z+1) + \omega_c^3 (z-1)^3}$$

$$= \frac{z^3 - 3z^2 + 3z - 1}{(\omega_c^3 + 2\omega_c^2 + 2\omega_c + 1)z^3 + (-3\omega_c^3 + 2\omega_c - 2\omega_c^2 + 1)z^2 + (3\omega_c^3 - 2\omega_c^2 - 2\omega_c + 1)z + (-\omega_c^3 + 2\omega_c^2 - 2\omega_c + 1)}$$

then substitute $\omega_c = 0.4426$

#

Q4-d)

For a 4th order BSF, we need 2nd order HPF

⇒ Bessel, $n=2$, $\omega_c = 50$, $B = 10$

$$G_{\text{HPF}}(s) = \frac{\Theta_n(s)}{\Theta_n(s/\omega_c)}$$

$$\Theta_n(x) = \sum_{k=0}^n a_k x^k, \quad \Theta_n(0) = \lim_{x \rightarrow 0} \Theta_n(x)$$

$$a_k = \frac{(2n-k)!}{2^{n-k} k! (n-k)!} = \frac{(4-k)!}{2^{4-k} k! (2-k)!}$$

$$\Rightarrow a_0 = \frac{4!}{2^4 0! 2!} = \frac{4 \times 3 \times 2}{4 \times 2} = 3$$

$$a_1 = \frac{3!}{2^3 1! 1!} = \frac{3 \times 2}{2} = 3$$

$$a_2 = \frac{2!}{2^2 2! 0!} = 1$$

$$\therefore \Theta_n(s/\omega_c) = \sum_{k=0}^2 a_k \left(\frac{s}{\omega_c}\right)^k$$

$$= 3 + 3s/\omega_c + s^2/\omega_c^2$$

$$\Theta_n(0) = \lim_{x \rightarrow 0} (3 + 3x + 3x^2) = 3$$

$$G_{\text{LPF}}(s) = \frac{3}{3 + 3s/\omega_c + 3s^2/\omega_c^2}$$

$$= \frac{3\omega_c^2}{3s^2 + 3\omega_c s + 3\omega_c^2}$$

$$G_{\text{HPF}}(s) = G_{\text{LPF}}\left(\frac{1}{s}\right) = \frac{3\omega_c^2}{3/s^2 + 3\omega_c/s + 3\omega_c^2}$$

$$= \frac{3\omega_c^2 s^2}{3\omega_c^2 s^2 + 3\omega_c s + 3}$$

$$G_{\text{BPF}}(s) = G_{\text{HPF}}\left(\frac{s^2 + \omega_c^2}{Bs}\right) = \frac{3\omega_c^2 \left(\frac{s^2 + \omega_c^2}{Bs}\right)^2}{3\omega_c^2 \left(\frac{s^2 + \omega_c^2}{Bs}\right)^2 + 3\omega_c \left(\frac{s^2 + \omega_c^2}{Bs}\right) + 3}$$

$$= \frac{3\omega_c^2 (s^4 + 2\omega_c^2 s^2 + \omega_c^4)}{3\omega_c^2 (s^4 + 2\omega_c^2 s^2 + \omega_c^4) + 3B\omega_c s (s^2 + \omega_c^2) + 3B^2 s^2}$$

$$= \frac{3\omega_c^2 (s^4 + 2\omega_c^2 s^2 + \omega_c^4)}{3\omega_c^2 s^4 + 3B\omega_c s^3 + (6\omega_c^4 + 3B^2)s^2 + 3B\omega_c^3 s + 3\omega_c^6}$$

then substitute with ω_c & B

##